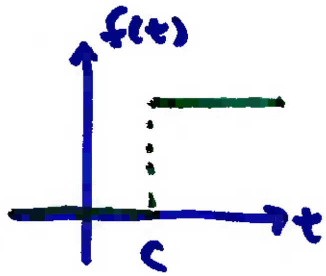
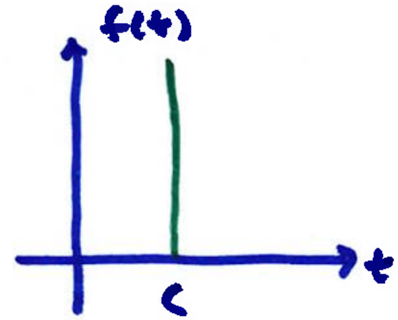
7.6 (continued)

$u_c(t)$



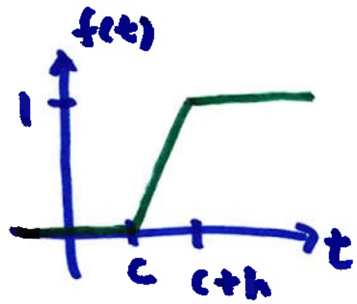
$\delta(t-c)$



it turns out $\frac{d}{dt} u_c(t) = \delta(t-c)$

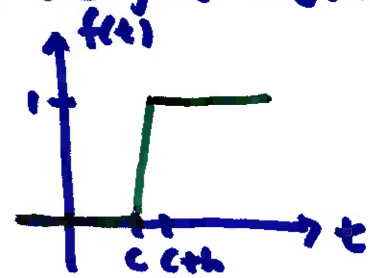
why?

ramp function

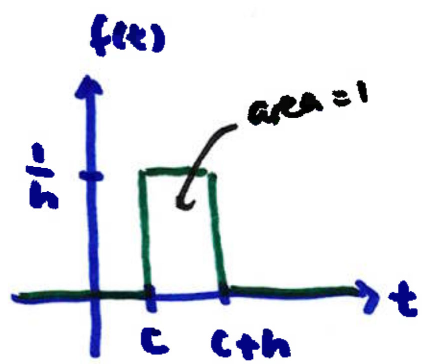


$h \to 0$ we get $u_c(t)$

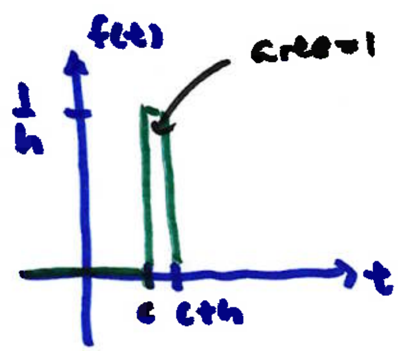
$\xrightarrow{h \to 0}$



derivative



$\xrightarrow{h \to 0}$



become $\delta(t-c)$

what $f(t)$ has Laplace transform of 1?

$$\mathcal{L}\{\delta(t-c)\} = e^{-cs}$$

if $c=0$, $\mathcal{L}\{\delta(t)\} = 1$ impulse at $t=0$

mass-spring-damper $ay'' + by' + cy = \delta(t)$ $y(0)=y'(0)=0$

$$(as^2+bs+c)Y = 1$$

$$Y = \frac{1}{as^2+bs+c}$$ impulse response of the system

$$y = \int_0^t g(t-\tau)\,\delta(\tau)\,d\tau$$

$ay'' + by' + cy = f(t)$ $y(0)=y'(0)=0$

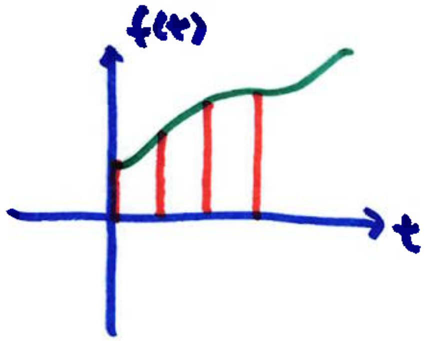
$$(as^2+bs+c)Y = F$$

$$Y = \underbrace{\frac{1}{as^2+bs+c}}_{\text{impulse response}} F$$

back to t: $y(t) = \int_0^t g(t-\tau)\,f(\tau)\,d\tau$ convolution

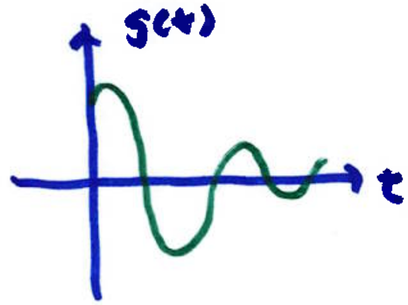
convolution is treating $f(t)$ as a bunch of impulses at different time

collect impulse responses at various t and stack them

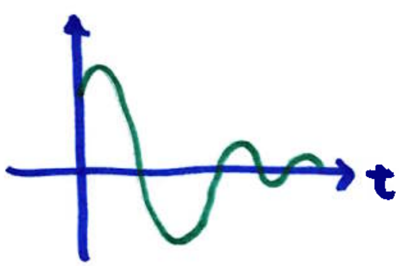


input

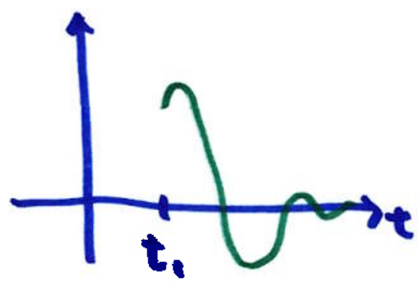
System impulse response is $g(t)$



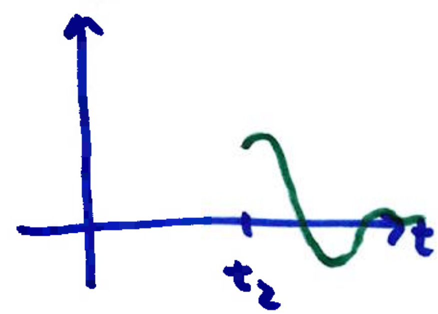
impulse at $t=0$



$t = t_1$

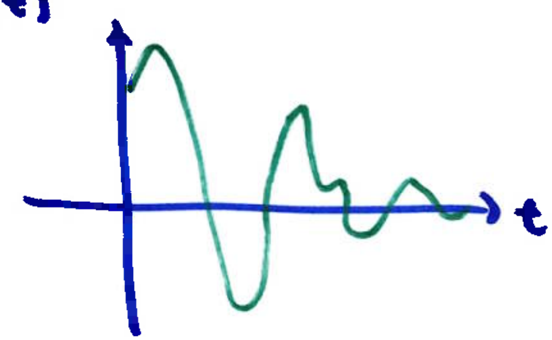


$t = t_2$



$t = t_3$ etc

Stack them → actual $y(t)$



impulse response

$$y = \int_0^t g(t-\tau)\, f(\tau)\, d\tau$$

force input

$$Y = G\,F$$

$$Y = \frac{G}{s} \cdot Fs$$

using some algebra and Laplace properties

$$y = h(t)\, f(0) + \int_0^t h(t-\tau)\, f'(\tau)\, d\tau$$

~~~~ step response

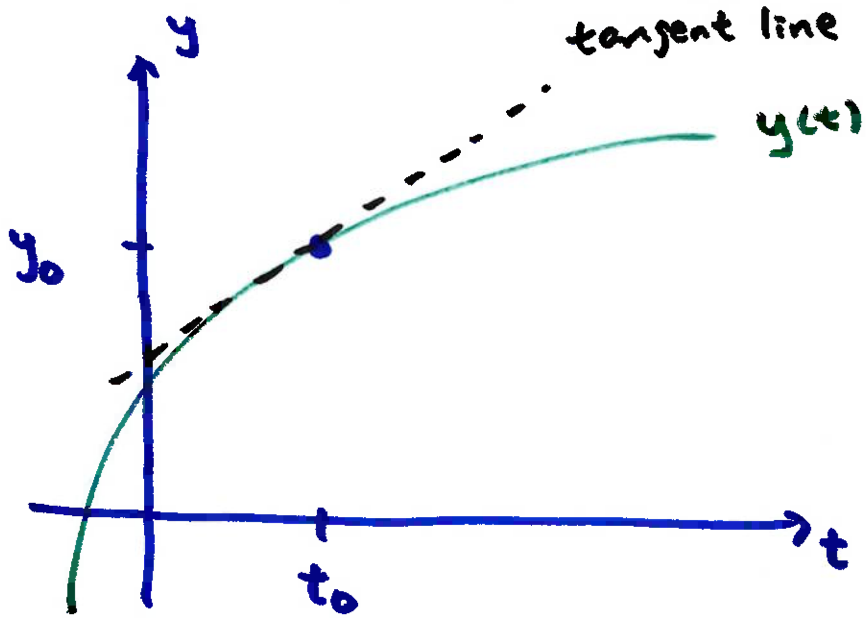
deriv. of force

interchangeable → Duhamel's Principle
~~~~

2.4 Euler's Method

numerical method to solve $y' = f(t,y)$ also applicable for higher order

basic idea: use tangent ~~ap~~ line approx. (like in calculus I)



$y(t)$ (unknown) but we have its slope at all (t,y)

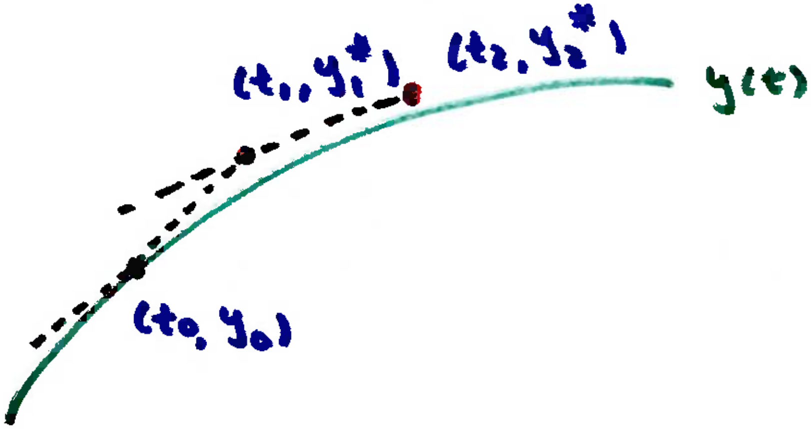
$y' = \underline{\underline{f(t,y)}}$

one point we know (t_0, y_0)

tangent line: $y - y_0 = \underbrace{f(t_0, y_0)}_{\text{slope from diff. eq.}}(t - t_0)$

$$y = y_0 + f(t_0, y_0)(t - t_0)$$

if $t - t_0$ is "small" this should be a "good" approx.



y_1^* is approx. y_1

y_2^* is approx. y_2

by doing another tangent line approx.

repeat until we reach target t

we can choose how far to move : $t - t_0 = h$ "step size"

<u>example</u> $y' = 2y - 3t \quad y(0) = 1$

use step size of $h = 0.25$ to estimate $y(0.5)$

two steps: $t_0 = 0$, target $t = 0.5$, step $= 0.25$

$$\left.\begin{array}{l} t_0 = 0 \\ y_0 = 1 \end{array}\right\} \text{given}$$

$$t_1 = t_0 + h = 0 + 0.25 = 0.25$$

$$y_1 = y_0 + \underbrace{f(t_0, y_0)}_{\text{slope at previous point}} (t_1 - t_0) = 1 + \left[2(1) - 3(0)\right](0.25) = 1.5$$

$t_2 = t_1 + h = 0.25 + 0.25 = 0.5$ target t, stop at end of this

$y_2 = y_1 + f(t_1, y_1)(t_2 - t_1) = \cdots = 2.0625$ approx.

accuracy? $y' = 2y - 3t \quad y(0) = 1$

solve $\cdots$ $y = \frac{3}{4}(2t+1) + \frac{1}{4}e^{2t}$ $\quad y(0.5) = 2.1796$

true

if $h = 0.01$ (50 steps)

$y \approx 2.1729$

Euler's method: error is proportional to h